Abstract

An exploration in the first half of this dissertation of the relationships among spectral sequences, persistent homology, and products of simplices, including the development of a new concept in categorical product filtration, is followed in the second half by new determinations of a) lower bounds for the Gromov-Hausdorff distance between \( n \)-spheres and \( (n+1) \)-hypercubes equipped with the geodesic metric and of b) new lower bounds for the coindexes of the Vietoris-Rips complexes of hypercubes equipped with the Hamming metric. In their paper, “Spectral Sequences, Exact Couples, and Persistent Homology of Filtrations” [?], Basu and Parida worked on building an \( n \)-derived exact couple from an increasing filtration \( X \) of simplicial complexes, \( C^{(n)}(X) = \{D^{(n)}(X), E^{(n)}(X), i^{(n)}, j^{(n)}, \partial^{(n)} \} \). The terms \( E^{(n)}_{*,*}(X) \) are the bigraded vector spaces of a spectral sequence that has differentials \( d^{(n)}(X) \), and the terms \( D^{(n)}_{*,*}(X) \) are the persistent homology groups \( H^{*,*}(X) \). They proved that there exists a long exact sequence whose groups are \( H^{*,*}(X) \) and whose bigraded vector spaces are \( (E^{*,*}(X), d^{*}(X)) \). We establish in Section ?? of this dissertation a new, similar
theorem in the case of the categorical product filtration $X \times Y$ that states that there exists a long exact sequence consisting of $\bigoplus_{l+j=n} H^*_l(X) \otimes H^*_j(Y)$ and of the bigraded vector spaces $E^*_*(X \times Y)$ of $(E^*_*(X \times Y), d^*(X \times Y))$, and prove it in part using Künneth formulas on homology. The emphasis on product spaces continues in Section ??, where we establish new lower bounds for the Gromov-Hausdorff distance between $n$-spheres and $(n+1)$-hypercubes, $I^{n+1}$, when both are equipped with the geodesic distance. From these lower bounds, we conjecture new lower bounds for the coindices of the Vietoris-Rips complexes of hypercubes when equipped with the Hamming metric.

We then determine new lower bounds for the coindices of the Vietoris-Rips complexes of hypercubes, a) by producing a map between spheres and the geometric realizations of Vietoris-Rips complexes of hypercubes using abstract convex combination and balanced sets, and b) by decomposing hollow $n$-cubes (homotopically equivalent to the above-mentioned spheres) into simplices of smaller dimension and smaller diameter.